Developing Mathematical Fluency

A Strategy to Help Children Learn Their Multiplication Facts

by Tracy L. Solomon and John Mighton

Mathematical fluency, also known as math or fact fluency, refers to the ability to recall answers to simple mathematical computation problems automatically, such as 2 + 3 = 5, 7 – 4 = 3, 5 x 6 = 30 and 42 ÷ 7 = 6. An important advantage of math fluency is that it frees up mental resources to focus on the deeper or conceptual aspects of the mathematics to be learned. In problem solving, for example, a child can focus on what the problem is about and on how to go about solving it, using his or her knowledge of math facts to assist in arriving at the problem solution. Indeed, performance on standardized measures of math fluency is robustly, positively correlated with performance on standardized measures of applied problem solving (McGrew & Woodcock, 2001). Historically, acquiring math fluency has largely involved drill. Many parents will remember using flashcards or performing oral drills as children to learn their multiplication tables. Unfortunately, such practices often fail to engage children's attention, which can lead to disinterest, or worse yet, to avoiding doing mathematics altogether.

Fortunately, there are more enjoyable ways to help children acquire math fluency. Below we briefly review key findings from the cognitive and neuroscience research on working memory, math fluency, and mathematics achievement. We then outline an engaging way to learn the multiplication or times tables that takes the research evidence into account. We focus on the times tables because it is critical to understanding more advanced concepts such as long division, fractions, decimals, percentages, and proportion. Consider how challenging it would be to find a common denominator when adding fractions or to do long division without a sound command of multiplication facts. We have used this approach, independently, and in diverse settings, with great success. We share it here in the hopes that educators, parents, and yet more children will profit from it.

The benefit of acquiring math fluency is that mathematical facts become automatized and stored in long-term memory, freeing up working memory resources to attend to deeper or more conceptual aspects of mathematics.

Brief Review of the Research

To understand how fluency is related to greater success at learning mathematics, it is useful to distinguish between working memory and long-term memory. Working memory refers to our ability to hold information in mind and to operate on that information, while long-term memory refers to our knowledge store. Working memory is engaged when, for example, you mentally carry out a two-digit addition problem that involves carrying. You have to calculate and then keep in mind the total in the ones column, as well as the number you are carrying to the tens column, while you compute the total in the tens column, and then remember what digits belong in which place in your final answer. Long-term memory is where you store facts you have learned such as all the factors that can be multiplied to arrive at the product of 24 (1, 2, 3, 4, 6, 8, 12 and 24). Working memory capacity is limited, especially so in children, and tasks that load up the learner with too much information can quickly overwhelm working memory, leaving few resources to attend to the material to be learned (Cowan, Morey, AuBuchon, Zwilling, & Gilchrist, 2010; Miller, 1956; Sweller & Chandler, 1994). In contrast, our long-term memory capacity is vast, possibly limitless. The benefit of acquiring math fluency is that mathematical facts become automatized and stored in long-term memory, freeing up working memory resources to attend to the deeper or more conceptual aspects of mathematics.

There is a good deal of evidence linking working memory with mathematics achievement. Several studies have shown that individual differences in working memory are related to individual differences in mathematics achievement in normally developing children (e.g., Bull & Scerif, 2001; Gathercole & Pickering, 2000; Holmes & Adams, 2006), and that difficulties in mathematics are related to poor working memory in children with mathematical disabilities (e.g. Gathercole & Pickering, 2000; Geary, Hamson, & Hoard, 2000). A more recent study that followed the same children through grades 1 and 2 showed that working memory assessed at the beginning of grade 1 actually predicted math achievement in the middle of grade 1 and at also the start of grade 2 (DeSmedt, Janssen, Bouwens, Verschaffel, Boets, & Ghesquiere, 2009). Still other research has shown that normally developing children progress from solving simple calculation problems by using procedures such as algorithms (which taxes working memory) to retrieving the answers from long-term memory (Geary, Brown, & Samaranayake, 1991), and that children with mathematical learning difficulties are delayed in this progression, if it occurs at all (Geary, 1993).

These findings are corroborated by the results from neuroscience research indicating age-related changes in the areas of the brain involved in processing simple arithmetic facts. When required to carry out single digit addition and subtraction problems, younger children show more activation in areas of the brain associated with working memory and attention while older children show more activation in areas associated with fact retrieval (Rivera, Reiss, Eckert, & Menon, 2005). Another study with grade 12 students examined areas of neural activation while performing single digit arithmetic and connected

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these areas of activation to performance on the quantitative subtest of a pre-college standardized achievement test the participants completed when they were in grade 10. Higher scores on the quantitative achievement test were significantly correlated with more activation in areas associated with fact retrieval and lower scores were significantly correlated with more activation in the area associated with number processing (Price, Mazzocco, & Ansari, 2013). One way to think about these findings is that the better your math fluency the more efficiently you use your brain, and the better your mathematics achievement. The question then is how to help children to acquire math fluency, in a way that engages their attention without overwhelming their working memory.

Emphasizing the conceptual basis of multiplication helps children acquire a deep understanding of multiplication but it is also important to find engaging ways for them to commit the multiplication facts to memory.

A Pattern Approach to Learning the Times Tables

We suggest that instruction that emphasizes the conceptual basis of multiplication is important for helping children to acquire a deep understanding of multiplication but that it is also important to find engaging ways for children to commit the multiplication facts to memory. The pattern approach to learning the multiplication times tables described here was developed by Dr. John Mighton, co-author of this article, a mathematician and the founder of the JUMP Math program of mathematics instruction (Mighton, Sabourin, & Klebanov, 2009; see *www.jumpmath.org*). This method capitalizes on our natural affinity for patterns, which is already present in infancy (see e.g., Humphrey, Humphrey, Muir, & Dodwell, 1986), to help children appreciate the notion of groupings—e.g., the idea that 2 x 3 means 2 groups, with 3 items in each group—and to use this knowledge quickly to generate the answers to the multiplication facts. Guiding children to appreciate the patterns in the different times tables in this way provides a conceptual basis on which to map their multiplication facts and, with practice, to be able to recall them automatically. We have found that children are highly enthusiastic at discovering the patterns in the various times tables and keen to demonstrate their new knowledge as they acquire it.

This method of learning the times tables is suitable for children in grade 3 (8 years of age) and up. It is recommended that children work with an adult for 10 to 15 minutes per day to learn their multiplication facts. Experience in diverse educational settings indicates that children as young as grade 3 can acquire fluency in about four weeks, but note that we do not recommend setting a time limit. Progress should follow the pace at which children are able to demonstrate mastery at each step. It is important to limit the amount of information children are required to learn at each step in the process to avoid overwhelming their working memory. If children are having difficulty, break down the information to be learned into even smaller chunks or sub-steps (some examples are provided below). Frequent assessment to ensure children have mastered each step is essential before moving on to new material.

Apart from the final two steps (5 and 6), the order of the steps is not fixed. We have used the order outlined below with great success. However, we have also tried starting with the 9x table because the patterns in the 9x table are very salient to children. The excitement they experience at learning the 9x table quickly can be highly motivating.

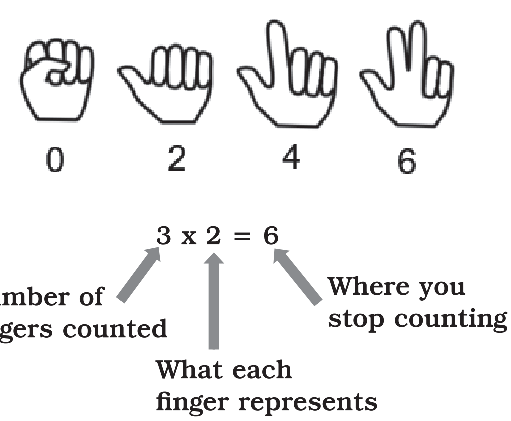
Steps

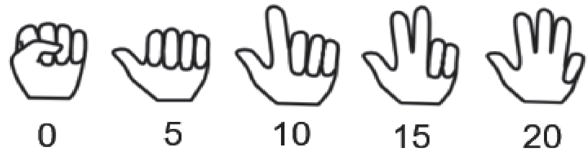
1. **2s and 5s**

   If children have not mastered their 2 and 5 times tables, begin by spending five to 10 minutes every day practicing skip counting by 2s, using their fingers (see the figures below). Show the child a closed fist, raise your thumb, then each finger in order saying the multiples of 2 aloud as you do; i.e., "2," "4," "6" and so on. Explain that each finger indicates that you counted 2 things and that the number of fingers raised when you stop counting shows how many groups of 2 you counted. Practice. Each time they stop counting check that they know how many things each finger means and how many groups of 2 they just counted.

   Connect this to a written multiplication sentence. For example, write "3 x 2" and then explain that the first number shows the number of fingers or groups you counted and the second number is the number of things in each group (i.e., that you are counting by 2s). The last number they say is the answer to the multiplication question. Continue until children can multiply up to 2 x 10 with ease.

   Repeat the process with 5s. Be sure to connect the skip counting on their fingers to the idea of groupings and to the written multiplication sentence as with the 2s. Practice until children can multiply up to 5 x 10 without hesitation.





4 x 5 = 20

Number of fingers counted

What each finger represents

Where you stop counting

2. 3s

Draw the 3 x 3 chart as shown below. Ask children if they see any patterns in the chart. Going across the rows, the number increases by 3. Looking at the columns, the number in the 10s column goes up by 1. The number in the 1s column goes down by 1.

Next, help children to memorize the chart. Provide a blank chart and fill in the top row. Have children copy the chart then fill in the columns using the patterns they just learned. If that proves too challenging, you could have them memorize just the first column, then the next column and so on until they can fill in the whole chart without hesitation.

Now connect this to multiplication. Tell children that every number has a position in the chart just like on a telephone pad. The number 3 is in position 1, the number 6 is in position 2, the number 9 in position 3, 12 in position 4 and so on. Practice identifying which number is in the different positions. For example, point to a number in the chart and ask what position it is in or cover up the chart, say a position number and ask children what number is there.

Now tell them that the number position in the chart is how many groups of 3 you have. The number 9 is in position 3, so you have 3 groups of 3, and 3 x 3 = 9. Connect this to the written multiplication statement as for 2s and 5s above. Repeat the process until children know the chart—or the 3x table—fluently.

03	06	09
12	15	18
21	24	27

3. 7s

Follow the same approach with the 7 times tables that you used for the 3 times tables. Looking across the rows, the number increases by 7. In the columns, the number in the 10s column increases by 2 and the number in the 1s column increases by 1. Work on recreating the chart, then on learning that the positions in the chart indicate the multiples of 7, then connect this to multiplication statements.

07	14	21
28	35	42
49	56	63

4. 9s

Write out the multiplication statements for 9 x 1 to 9 x 10 as shown here.

9x1 = 09
9x2 = 18
9x3 = 27
9x4 = 36
9x5 = 45
9x6 = 54
9x7 = 63
9x8 = 72
9x9 = 81
9x10 = 90

Ask students what patterns they see in the table. Help them as needed by pointing out that as you go down the answer column, the number in the 10s column increases by 1 and the number in the 1s column decreases by 1. Write the first fact: 9 x 1 = 09. Have children copy this and then generate the other facts in a vertical chart using the patterns they just learned. Have them practice until they can do this easily. On learning these patterns for the 9x table, one child spontaneously exclaimed that it makes sense because every time you have 9 more, it's like adding 10 and then taking 1 away. This shows a good conceptual understanding of the notion of groups of 9.

On learning patterns for the 9x table, one child spontaneously exclaimed that it makes sense because every time you have 9 more, it's like adding 10 and then taking 1 away. This shows good conceptual understanding.

Another way to approach learning the 9x table is to point out that the digits in the product (the answer) always sum to 9. For example, 3 x 9 = 27, and 2 + 7 = 9. Similarly, 5 x 9 = 45, and 4 + 5 = 9. Also, the number in the 10s column is always 1 less than the number you are multiplying by (this makes sense because you are multiplying by 1 less than 10). Using these two pattern rules, children can quickly generate the answer to any multiplication question in the 9x table. For example, for 9 x 4, the answer must begin with 3 (1 less than the number you are multiplying by). The number in the ones column is then 9 – 3, or 6. So 9 x 4 = 36.

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5. **Even numbers times 6**
   Write out the chart below. Point out the patterns as with the other times tables. In the answer column, the number in the 10s column increases by 1 and the number in the 1s column increases by 2. The number in the 1s column is also the same as the number you are multiplying by. Write the first multiplication sentence, have children copy it and then write the rest of the multiplication sentences in the chart. Practice until mastery.

2 x 6 = 12

4 x 6 = 24

6 x 6 = 36

8 x 6 = 48

6. **Remaining 3 facts**
   To learn the final 3 facts below, we suggest doubling—multiplying by 2 and then doubling the answer. For example, 4 x 2 = 8 and 8 x 2 (doubled) = 16. So 4 x 4 = 16.

4 x 4 = 16

4 x 8 = 32

8 x 8 = 64

Multiplying by 1 and 10 are not covered in the steps above. Although it may seem straightforward how to arrive at the correct answers for the 1 and 10 times tables, we suggest that working through some examples, even briefly, remains an important opportunity to review the idea of grouping with children—i.e., that 2 x 1 is 2 groups with 1 item in each group and likewise that 8 x 10 indicates 8 groups with 10 items in each group, which may help them to see that the principle of grouping applies to all of the multiplication tables.

The pattern approach to learning the multiplication times tables described here engages children's attention without overwhelming their working memory and could help them to acquire their multiplication facts quickly. Once learned, math facts are transferred to their long-term memory store and can be retrieved as needed to assist in doing mathematics, such as in problem solving. This frees up working memory resources, which can then be applied to the mathematical concepts to be learned. Knowing their math facts thus allows them to use their brain more efficiently, with considerable benefits for mathematics achievement.

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